A Method of

Estimating Cubic Volume

in Felled Trees

by Frederick E. Hampf & C. Allen Bickford

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About the Authors...

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THIS paper describes a new method of computing the gross volume of felled trees or sections of trees. The method has many applications, but most are limited to forest research and management, especially in making surveys.

In a sense, this method is a compromise between the tedious but fairly accurate planimeter method and the faster but less accurate method of estimating by diameter (d.b.h.) volume tables. We feel that the method detailed here is twice as fast as measuring by planimeter and far more accurate than estimating by whole-tree volume tables.

It offers less chance of errors in human judgment and can mean fewer office computations. It has the easy practicality that foresters require, while approximating the accurate results--butt logs notwithstanding--of the xylometer, use of which is highly restricted.

This method is being used in wood-utilization studies for the Forest Survey in the Northeast. The Forest Survey reports the volume of timber cut from forests within a given state. This volume, expressed in cubic feet, includes timber cut and removed and also logging residues left in the woods, including trees destroyed by logging. These differences--between timber actually harvested and Forest Survey utilization standards--must also be determined to estimate the volume of timber cut.

Assumptions of Tree Form

To compute the total wood content of a tree one must assume that a tree is made up of certain geometric forms. For simplicity, we assume that the stump is a cylinder (fig. 1) with a diameter equal to its top diameter, inside bark, (average level or horizontal plane), and with a height of 1 foot from ground level (uphill side of tree).

We assume that the tree bole from 1 foot to 9.15 feet forms a frustum of a neiloid. Above this point the taper rate increases with height. Therefore, the upper bole, from 9.15 feet to the base of the crown, usually takes the general geometric form of the frustum of a paraboloid. We further assume that the top of fast-tapering single-stem trees forms the frustum of a cone.

Trees of unusual shape require special consideration. Each portion of such a tree should be fitted as circumstances require to the frustum of either a neiloid, paraboloid, cone, or cylinder.

Based on these assumptions, the measurements needed to compute total wood volume in a tree are diameter measurements at 1.00, 5.075, and 9.15 feet above ground level, at the upper end of each bucked section, and at the flexible 4-inch top.[1] Length measurements are needed for each section, including the stump and the portion of the top used above (or left below) the 4-inch top. Diameter measurements are taken to 0.1 inch, length to the nearest 0.1 foot.

Computations Required

Volumes of each of the log sections can be computed as follows:

Stump.--For Forest Survey purposes, only the volume of the stump between the assumed 1-foot height and actual stump height is necessary. Wood is "over-utilized" when a stump is cut below the 1-foot height, "under-utilized" when cut above it. Stump volume is obtained by multiplying the area (Appendix I, table 1) of the stump's top by the difference in height.

Butt log.--The lower 8-foot section of the 16-foot butt log is in the form of a neiloid frustum, and the upper portion of the butt log is in the form of a paraboloid frustum. Thus two steps are required to compute butt-log volume. Diameter measurements are needed at 1.00, 5.075, and 9.15 feet above ground level and at the top of the log. Volume of the butt 8-foot section is obtained from Circular Slide Rule A (Appendix II). Volume for the upper portion is obtained from Circular Slide Rule B (Appendix III), using diameters at 9.15 feet and log top.

Upper logs.--The cubic volume of 16-foot upper logs is obtained directly from Circular Slide Rule B. The volume of upper logs shorter than 16 feet is obtained as if they were 16-foot logs--then an appropriate reduction factor is applied. Reduction factors are given on the reverse side of Circular Slide Rule B.

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[1] To an obvious 4-inch top (d.i.b.) or to the point at which the main stem disappears because of branching.

Upper Stem.--The volume of over- or under-utilization (Forest Survey standards) of the upper stem is also required. This is the departure in cubic volume from the "flexible 4-inch top".

The upper stem in single-stem trees, which taper rapidly (no large and excessive branching) to an obvious 4-inch top d.i.b., generally approximates the form of a frustum of a cone.

The cubic volume for this portion of the tree is obtained from table 2 (Appendix IV). Greater accuracy may be got by subdividing any length in excess of 16 feet. The volume of this "butt" section is obtained from Circular Slide Rule B; the volume for the upper section is obtained from table 2.

Whenever abrupt taper occurs at localized points along the upper stem because of large or excessive branching, each section between these points should be considered as a frustum of a paraboloid, and its volume may be obtained from Circular Slide Rule B.

Limbs.--The cubic volume of all limbs utilized is also determined. Limb form is not parabolic but is assumed to be so for convenience; and limb volume, like that of upper logs, is obtained from Circular Slide Rule B. The limb's "butt" diameter should be measured away from and just below the point of noticeable knot swell. Once computed, this volume is over-utilization.

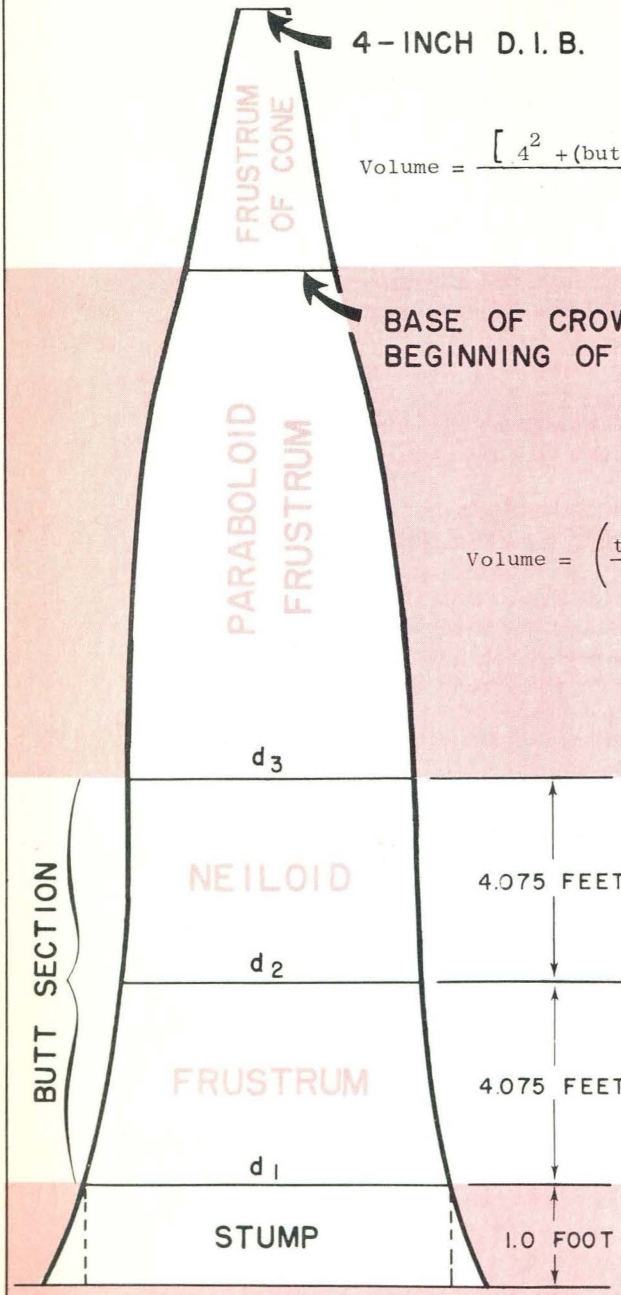
Example of Procedure

The following example shows how to compute the cubic volume for (1) the entire portion of a tree cut and removed and (2) the unutilized portion left in the woods. This computation is based on Forest Survey standards.

Measurements
Of The Felled Tree

The tree was felled, leaving a stump 1.3 feet high and 25.4 inches in diameter. The tree's diameter measurements (inside bark) were:

	Inches
at 1.00 foot --	26.0
at 5.075 feet --	20.6
at 9.15 feet --	19.9



$$\text{Volume} = \frac{\left[\,4^2 + (\text{butt diam.})(\text{butt diam.} + 4)\right]}{3}\left(\frac{\pi}{576}\right)(L)$$

$$\text{Volume} = \left(\frac{\text{top diam.}^2 + \text{butt diam.}^2}{2}\right)\left(\frac{\pi}{576}\right)(L)$$

$$\text{Volume} = \frac{d_3(3d_3 + t_1 + 4t_2)}{3}\left(\frac{\pi L}{576}\right)$$

$$t_1 = d_1 - d_3$$

$$t_2 = d_2 - d_3$$

$$\text{Volume} = (\text{top area})(\text{stump height}) = \left(\frac{\pi d^2}{576}\right)(L)$$

The formula for determining the volume of a neiloid frustrum was derived from Newton's formula:

$$\text{Volume} = (B_1 + 4B_2 + B_3)\ \frac{L}{6}$$

$$= \left(\frac{d_1^2 + 4d_2^2 + d_3^2}{6}\right)\left(\frac{\pi L}{576}\right)$$

From the sketch:

$t_1 = d_1 - d_3$ or $d_1 = d_3 + t_1$ $\qquad \frac{\pi}{576} = \frac{3.1416}{576} = 0.005454...$

$t_2 = d_2 - d_3$ or $d_2 = d_3 + t_2$ $\qquad L = 8.15$ feet

Thus $d_1^2 + 4d_2^2 + d_3^2 = (d_3 + t_1)^2 + 4(d_3 + t_2)^2 + d_3^2$

$$= 6d_3^2 + 2d_3(t_1 + 4t_2) + t_1^2 + 4t_2^2$$

And, as $t_1^2 + 4t_2^2$ is negligible in comparison with $6d_3^2 + 2d_3(t_1 + 4t_2)$

Volume is approximately $\left(d_3^2 + \frac{d_3(t_1 + 4t_2)}{3}\right)\left(\frac{\pi L}{576}\right)$

Approximate volume of frustrum of neiloid $= \frac{d_3(3d_3 + t_1 + 4t_2)}{3}\left(\frac{\pi L}{576}\right)$

Figure 1.--Tree forms and the formulas used to compute their cubic volume.

Material cut and used:

	Length (feet)	Top diameter (inches)
Butt log	16.5	18.2
Second log	11.4	14.8
Third log	10.7	13.1
Fourth log	8.4	10.2

Material left in woods:

Tree top	19.8	4.0

(The diameter of this tree top, taken at approximately mid-section or at 10 feet from the butt, was 7.4 inches, and it tapered rapidly.)

Volume Of Material Cut And Removed

1

Determine stump volume. Use Table 1.

Stump volume = 3.519 square feet (area for 25.4-inch diameter)
x 0.3 feet (difference between 1.0 foot and actual 1.3-foot heights)

= 1.06 cubic feet (stump volume not harvested)

2

Determine volume of butt log. Use Circular Slide Rules A and B. Directions for use of the slide rules are printed on the movable wheel of each.

I. Determine the taper factor for butt 8-foot section.

t_1 = 26.0 (at 1.00 foot) - 19.9 (at 9.15 feet) = 6.1 inches

t_2 = 20.6 (at 5.075 feet) - 19.9 (at 9.15 feet) = 0.7 inch

$$\text{Taper factor} = (3d_3 + t_1 + 4t_2)$$

in which d_3 is the diameter at 9.15 feet;

$$3d_3 = 19.9 \times 3 = 59.7;$$

so taper factor = (59.7 + 6.1 + 2.8) = 68.6 inches.

II. Determine volume of butt 8-foot section:

20.2 (from Circular Slide Rule A
- 1.06 (stump volume not harvested)
= 19.14 cubic feet

III. Determine volume of upper portion of butt log:

Length of upper portion = 16.5 feet - 8.2 feet (butt section)
= 8.3 feet

Diameters of upper portion = 18.2 inches at top
19.9 inches at 9.15 feet

Volume as 16-foot log
(from Circular Slide Rule B) = 31.6 cubic feet

Volume for 8.3-foot log = 31.6 x 0.5188 (reduction factor from Circular Slide Rule B) = 16.39 cubic feet

IV. Volume of butt log = 19.14 + 16.39 = 35.53 cubic feet.

3

Determine volume of 2nd log.

Volume as a 16-foot log
(from Circular Slide Rule B) = 24.1 cubic feet

Volume for 11.4-foot log = 24.1 x 0.7125 = 17.17 cubic feet.

4

Determine volume of 3rd log.

Volume as a 16-foot log
(from Circular Slide Rule B) = 17.0 cubic feet.

Volume for 10.7-foot log = 17.0 x 0.6688 = 11.37 cubic feet.

5

Determine volume of 4th log.

Volume as a 16 foot log
(from Circular Slide Rule B) = 12.1 cubic feet.

Volume for 8.4-foot log = 12.1 x 0.5250 = 6.35 cubic feet.

6

Determine total volume cut and removed from tree.

35.53 + 17.17 + 11.37 + 6.35 = 70.42 cubic feet.

Volume Of Upper Stem
Left In The Woods

1

Volume of the upper stem "butt" section is obtained from Circular Slide Rule B, using diameters 10.2 and 7.4 inches and length 10.0 feet as follows:

Volume as 16-foot log = 6.6 cubic feet.

Volume for 10.0-foot log = 6.6 x 0.6250 = 4.12 cubic feet.

2

Determine volume of top's upper section (from table 2).

Interpolating:
volume for 7.4 inches x 10 feet long = 1.83 cubic feet.

Adjusting for length of 9.8 feet:
(1.83) (0.98) = 1.79 cubic feet.

3

Determine volume of top not utilized = 4.12 + 1.79
= 5.91 cubic feet.

APPENDIX

I

Table 1.--Areas of circles, in square feet, for diameters in tenths of inches

Diameter	Tenths of inches									
Inches	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
5	0.136	0.142	0.147	0.153	0.159	0.165	0.171	0.177	0.183	0.190
6	.196	.203	.210	.216	.233	.230	.238	.245	.252	.260
7	.267	.275	.283	.291	.299	.307	.315	.323	.332	.340
8	.349	.358	.367	.376	.385	.394	.403	.413	.422	.432
9	.442	.452	.462	.472	.482	.492	.503	.513	.524	.535
10	.545	.556	.567	.579	.590	.601	.613	.624	.636	.648
11	.660	.672	.684	.696	.709	.721	.734	.747	.759	.772
12	.785	.799	.812	.825	.839	.852	.866	.880	.894	.908
13	.922	.936	.950	.965	.979	.994	1.009	1.024	1.039	1.054
14	1.069	1.084	1.100	1.115	1.131	1.147	1.163	1.179	1.195	1.211
15	1.227	1.244	1.260	1.277	1.294	1.310	1.327	1.344	1.362	1.379
16	1.396	1.414	1.431	1.449	1.467	1.485	1.503	1.521	1.539	1.558
17	1.576	1.595	1.614	1.632	1.651	1.670	1.689	1.709	1.728	1.748
18	1.767	1.787	1.807	1.827	1.847	1.867	1.887	1.907	1.928	1.948
19	1.969	1.990	2.011	2.032	2.053	2.074	2.095	2.117	2.138	2.160
20	2.182	2.204	2.226	2.248	2.270	2.292	2.315	2.337	2.360	2.382
21	2.405	2.428	2.451	2.474	2.498	2.521	2.545	2.568	2.592	2.616
22	2.640	2.664	2.688	2.712	2.737	2.761	2.786	2.810	2.835	2.860
23	2.885	2.910	2.936	2.961	2.986	3.012	3.038	3.064	3.089	3.115
24	3.142	3.168	3.194	3.221	3.247	3.274	3.301	3.328	3.355	3.382
25	3.409	3.436	3.464	3.491	3.519	3.547	3.574	3.602	3.631	3.659
26	3.687	3.715	3.744	3.773	3.801	3.830	3.859	3.888	3.917	3.947
27	3.976	4.006	4.035	4.065	4.095	4.125	4.155	4.185	4.215	4.246
28	4.276	4.307	4.337	4.368	4.399	4.430	4.461	4.493	4.524	4.555
29	4.587	4.619	4.650	4.682	4.714	4.746	4.779	4.811	4.844	4.876
30	4.909	4.942	4.974	5.007	5.041	5.074	5.107	5.140	5.174	5.208
31	5.241	5.275	5.309	5.343	5.378	5.412	5.446	5.481	5.515	5.550
32	5.585	5.620	5.655	5.690	5.726	5.761	5.796	5.832	5.868	5.904
33	5.940	5.976	6.012	6.048	6.084	6.121	6.158	6.194	6.231	6.268
34	6.305	6.342	6.379	6.417	6.454	6.492	6.529	6.567	6.605	6.643
35	6.681	6.720	6.758	6.796	6.835	6.874	6.912	6.951	6.990	7.029
36	7.069	7.108	7.147	7.187	7.227	7.266	7.306	7.346	7.386	7.426
37	7.467	7.507	7.548	7.588	7.629	7.670	7.711	7.752	7.793	7.834
38	7.876	7.917	7.959	8.001	8.042	8.084	8.126	8.169	8.211	8.253
39	8.296	8.338	8.381	8.424	8.467	8.510	8.553	8.596	8.640	8.683
40	8.727	8.770	8.814	8.858	8.902	8.946	8.990	9.035	9.079	9.124
41	9.168	9.213	9.258	9.303	9.348	9.393	9.439	9.484	9.530	9.575
42	9.621	9.667	9.713	9.759	9.805	9.852	9.898	9.945	9.991	10.038
43	10.085	10.132	10.179	10.226	10.273	10.321	10.368	10.416	10.463	10.511
44	10.559	10.607	10.655	10.704	10.752	10.801	10.849	10.898	10.947	10.996
45	11.045	11.094	11.143	11.192	11.242	11.291	11.341	11.391	11.441	11.491
46	11.541	11.591	11.642	11.692	11.743	11.793	11.844	11.895	11.946	11.997
47	12.048	12.100	12.151	12.203	12.254	12.306	12.358	12.410	12.462	12.514
48	12.566	12.619	12.671	12.724	12.777	12.830	12.882	12.936	12.989	13.042
49	13.095	13.149	13.203	13.256	13.310	13.364	13.418	13.472	13.527	13.581
50	13.635	13.690	13.745	13.800	13.854	13.909	13.965	14.020	14.075	14.131

II

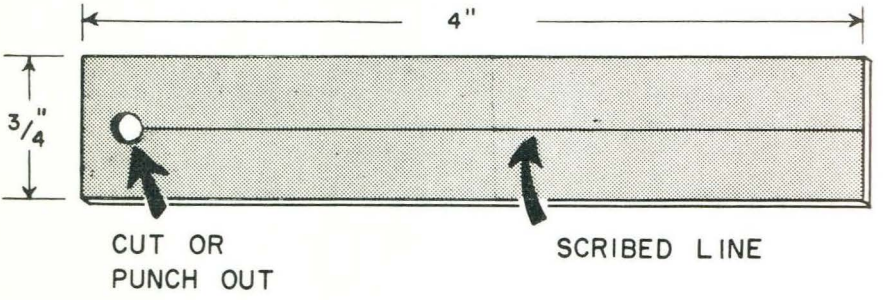
Circular Slide Rule A is used for determining cubic volume in 8-foot butt sections.

III

Circular Slide Rule B is used for determining cubic volume in upper logs, 16-foot length.

For convenience in use, both Slide Rules A and B are inserted unbound in this paper. Each consists of two parts, a base and a movable wheel. To assemble, cut out the movable wheel and fasten it to the base with a metal grommet, binder post, or pin. The centers are marked. The slide rules can be stiffened and made more durable by covering both parts with clear acetate.

Use of a finder made from a piece of stiff clear acetate will increase accuracy and reduce the time required to locate the volume opposite the desired butt-diameter reading. The diagram below can be used as a pattern for making such a finder.



The size of the hole punched or cut out of the finder will depend on the size of the grommet or pin used in assembling the parts of the circular slide rule.

Directions for use of the slide rules are printed on the movable wheel of each.

IV

Table 2.--Cubic-foot volume of single-stem tree tops

(Truncated at fixed 4-inch top d.i.b. only)

b = butt d.i.b.

Formula for cone frustrum: Vol. = $\frac{[4^2+b(b+4)]}{3}\left(\frac{\pi}{(4:144)}\right)L$

= $[16+b(b+4)]\ (0.001818)\ L$

Butt diameter (inches, inside bark)	Length (feet)												
	4	5	6	7	8	9	10	11	12	13	14	15	16
5.0	0.44	0.55	0.67	0.78	0.89	1.00	1.11	1.22	1.33	1.44	1.55	1.66	1.77
5.5	.50	.62	.74	.87	.99	1.12	1.24	1.36	1.49	1.61	1.74	1.86	1.99
6.0	.55	.69	.83	.97	1.11	1.24	1.38	1.52	1.66	1.80	1.93	2.07	2.21
6.5	.61	.77	.92	1.07	1.23	1.38	1.53	1.68	1.84	1.99	2.14	2.30	2.45
7.0	.68	.85	1.01	1.18	1.35	1.52	1.69	1.86	2.03	2.20	2.37	2.54	2.71
7.5	.74	.93	1.12	1.30	1.49	1.67	1.86	2.04	2.23	2.42	2.60	2.79	2.97
8.0	.81	1.02	1.22	1.43	1.63	1.83	2.04	2.24	2.44	2.65	2.85	3.05	3.26
8.5	.89	1.11	1.33	1.56	1.78	2.00	2.22	2.44	2.67	2.89	3.11	3.33	3.56
9.0	.97	1.21	1.45	1.69	1.93	2.18	2.42	2.66	2.90	3.14	3.39	3.63	3.87
9.5	1.05	1.31	1.57	1.84	2.10	2.36	2.62	2.88	3.15	3.41	3.67	3.93	4.20
10.0	1.13	1.42	1.70	1.99	2.27	2.55	2.84	3.12	3.40	3.69	3.97	4.25	4.54
10.5	1.22	1.53	1.84	2.14	2.45	2.75	3.06	3.36	3.67	3.98	4.28	4.59	4.89
11.0	1.32	1.65	1.97	2.30	2.63	2.96	3.29	3.62	3.95	4.28	4.61	4.94	5.26
11.5	1.41	1.77	2.12	2.47	2.83	3.18	3.53	3.88	4.24	4.59	4.94	5.30	5.65
12.0	1.51	1.89	2.27	2.65	3.03	3.40	3.78	4.16	4.54	4.92	5.29	5.67	6.05
12.5	1.62	2.02	2.42	2.83	3.23	3.64	4.04	4.44	4.85	5.25	5.66	6.06	6.46
13.0	1.72	2.15	2.59	3.02	3.45	3.88	4.31	4.74	5.17	5.60	6.03	6.46	6.89
13.5	1.83	2.29	2.75	3.21	3.67	4.13	4.59	5.04	5.50	5.96	6.42	6.88	7.34
14.0	1.95	2.44	2.92	3.41	3.90	4.39	4.87	5.36	5.85	6.33	6.82	7.31	7.80
14.5	2.07	2.58	3.10	3.62	4.13	4.65	5.17	5.68	6.20	6.72	7.23	7.75	8.27
15.0	2.19	2.74	3.28	3.83	4.38	4.92	5.47	6.02	6.57	7.11	7.66	8.21	8.76
15.5	2.31	2.89	3.47	4.05	4.63	5.21	5.79	6.36	6.94	7.52	8.10	8.68	9.26
16.0	2.44	3.05	3.67	4.28	4.89	5.50	6.11	6.72	7.33	7.94	8.55	9.16	9.77
16.5	2.58	3.22	3.86	4.51	5.15	5.80	6.44	7.08	7.73	8.37	9.02	9.66	10.30
17.0	2.71	3.39	4.07	4.75	5.42	6.10	6.78	7.46	8.14	8.82	9.49	10.17	10.85
17.5	2.85	3.57	4.28	4.99	5.70	6.42	7.13	7.84	8.56	9.27	9.98	10.70	11.41
18.0	3.00	3.75	4.49	5.24	5.99	6.74	7.49	8.24	8.99	9.74	10.49	11.24	11.98

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